Korean J. of Chem. Eng., **13**(2), 165-171 (1996)

BUOYANCY-DRIVEN CONVECTION IN A HORIZONTAL FLUID LAYER UNDER UNIFORM VOLUMETRIC HEAT SOURCES

Min Chan Kim*, Do-Young Yoon† and Chang Kyun Choi**

*Department of Chemical Engineering, Cheju National University, Cheju 690-756, Korea
Department of Chemical Engineering, Institute of New Technology, Kwangwoon University, Seoul 139-701, Korea
**Department of Chemical Engineering, Seoul National University, Seoul 151-742, Korea
(Received 20 September 1995 • accepted 10 January 1996)

Abstract—Buoyancy effect in an internally heated horizontal fluid layer is considered under the linear stability analysis. The horizontal fluid layer is confined between a rigid adiabatic lower boundary and a rigid isothermal upper boundary. The onset of thermal convection is analyzed by using the propagation theory which transforms partial disturbance equations into ordinary ones similarly under the principle of exchanges of stabilities. The eigenvalue problem is solved by the method of rapidly converging power series. In addition, the connection of stability condition to the fully developed heat transport is investigated. Results show that the critical time to mark cellular convection has increased with a decrease in the Prandtl number. Based on the present stability criteria, a new correlation of the Nusselt number is produced as a function of both the Rayleigh number and the Prandtl number. It is shown that the present correlation on thermal convection compares reasonably with existing experimental data of water.

Key words: Buoyancy Effects, Propagation Theory, Nusselt Number, Prandtl Number, Rayleigh Number

INTRODUCTION

From the beginning of this century the convective motion driven by buoyancy forces has attracted many researchers' interests. Benard [1901] conducted systematic experiments on the onset of natural convection in a horizontal fluid layer. Later, Lord Rayleigh [1916] showed that the buoyancy-driven convection can occur when the adverse temperature gradient exceeds a certain critical value. Thereafter, many researchers analyzed the onset condition of buoyancy driven convection in fluid layers heated from below or cooled from above. Extensive results for the various systems have been summarized by Chandrasekhar [1961] and Berg et al. [1974].

Kulacki and Goldstein [1975] extended the stability analysis to the horizontal fluid layer heated by internal heat sources. It is well-known that thermal convection problems driven by energy release from distributed volumetric energy sources appear to play an important role in wide variety of engineering applications, such as geothermal reservoirs, chemical reactors and heat removal of nuclear power plants.

When an initially quiescent horizontal fluid layer system is heated rapidly, buoyancy-driven motion sets in before the basic temperature field is fully-developed. Therefore, in case of rapid heating the basic temperature profile of pure conduction becomes time-dependent. To analyze this kind of thermal instability in horizontal fluid layers several theoretical methods have been proposed: the amplification theory [Foster, 1965], the energy method [Wankat and Homsy, 1977], the stochastic model [Jhavery and Homsy, 1982] and the propagation theory [Choi et al., 1984]. The amplification theory treated the time dependency as an initial value problem. This method is quite popular, but it involves arbitrariness in choosing both an initial condition and its amplification factor to mark the onset of motion. The propagation theory predicts the conditions to mark the onset time deterministically, it employs the thermal penetration depth as a length scaling factor and transforms the linearized disturbance equations into the similar forms. Its prediction has been coincident with the various experimental results in deep-pool systems experiencing rapid heating, such as laminar forced convection [Kim et al., 1990; Choi and Kim, 1990], laminar natural convection [Chun and Choi, 1991] and also double-diffusive convection [Yoon et al. 1995].

Another important problem in buoyancy-driven phenomena is the heat transfer characteristics in thermally fully-developed state. To analyze this problem Howard [1964] proposed the boundary-layer instability model in which the heat transfer for very high Rayleigh numbers has a close relationship with stability criteria. Based on Howard's concept, Long [1976] and Cheung [1980] introduced backbone equations to predict the heat transport in horizontal fluid layers. By incorporating their stability criteria into the boundary layer instability model Choi et al. [1989] and Kim and Choi [1992] have derived new heat transfer correlations for various systems. Based on the microscales of turbulent flow Arpaci [1994] proposed a new buoyancy-driven heat transfer model for fully developed-turbulent state.

In the present study, the stability criteria of the onset of regular cell-type motion in a horizontal fluid layer with uniform energy sources is analyzed by using our propagation theory. And based on the stability criteria and Arpaci's heat transfer model, a new heat transfer correlation is derived and also compared with the existing experimental results. This research shows that the propagation theory we have developed can become a theoretical base in understanding buoyancy-driven phenomena.

†To whom all correspondences should be addressed.

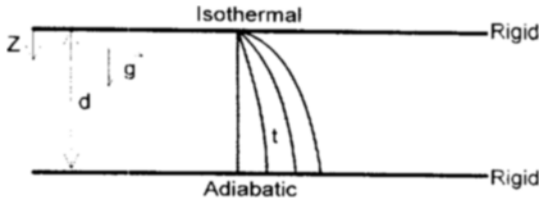

Fig. 1. Schematic diagram of the present system.

STABILITY ANALYSIS

1. Governing Equations

The system considered here is an initially quiescent horizontal fluid layer of depth "d" with an adiabatic lower boundary and isothermal upper boundary. Before heating the fluid layer is maintained at uniform temperature T_0 for time $t<0$. For time $t\geq0$ the layer is heated internally with the uniform volumetric heat generation rate S. Here we employ the Cartesian coordinates with the downward distance Z. The schematic diagram of present system is shown in Fig. 1. In this system the governing equations of flow and temperature fields are expressed by employing the Boussinesq approximation, as follows:

$$\nabla\cdot\vec{U}=0 \tag{1}$$

$$\left\{\frac{\partial}{\partial t}+\vec{U}\cdot\nabla\right\}\vec{U}=-\frac{1}{\rho_r}\nabla P+\nu\nabla^2\vec{U}+g\beta T\vec{k} \tag{2}$$

$$\left\{\frac{\partial}{\partial t}+\vec{U}\cdot\nabla\right\}T=\alpha\nabla^2 T+\frac{S}{\rho_r C_p} \tag{3}$$

where $\vec{U}$ is the velocity vector, T the temperature, P the pressure, μ the viscosity, α the thermal diffusivity, g the gravitational acceleration, ρ the density, C_p the specific heat, β the thermal expansion coefficient. The subscript "r" represents the reference state.

The important parameters to describe the present system are the Prandtl number Pr and the Rayleigh number Ra_I defined by

$$Pr=\frac{\nu}{\alpha} \text{ and } Ra_I=\frac{g\beta S d^5}{k\alpha\nu} \tag{4}$$

where k and ν denote the thermal conductivity and the kinematic viscosity, respectively. In case of slow heating the basic temperature profile is parabolic and time-independent and its critical condition is well represented by

$$Ra_{I,c}=2,772 \tag{5}$$

But for a rapid heating system of large Ra_I, the stability problem becomes transient and complicated, and the critical time t_c to mark the onset of buoyancy-driven motion remains unsolved. For this transient stability analysis we define a set of nondimensional variables τ, z, θ_0 by using the scale of time d^2/α, length d and temperature Sd^2/k. Then the basic conduction state is represented in dimensionless form by

$$\frac{\partial\theta_0}{\partial\tau}=\frac{\partial^2\theta_0}{\partial z^2}+1 \tag{6}$$

with the following initial and boundary conditions,

$$\theta_0(0,\ z)=\theta_0(\tau,\ 0)=\frac{\partial\theta_0}{\partial z}(\tau,\ 1)=0 \tag{7}$$

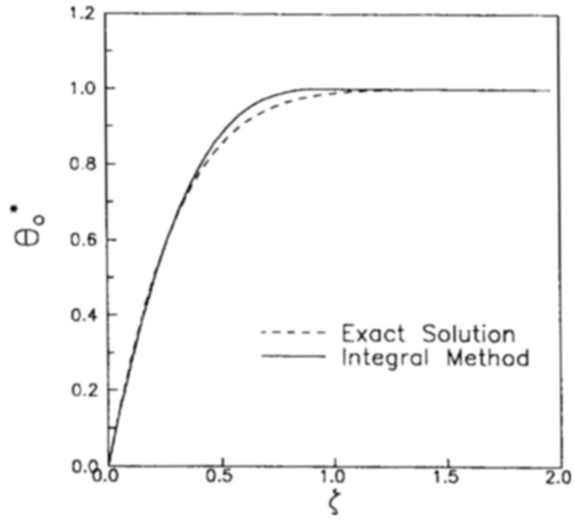

Fig. 2. Basic temperature profile.

By using the conventional separation-of-variables technique, the above conduction equation can be solved as follows:

$$\theta_0=z\left(1-\frac{z}{2}\right)-\frac{16}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\sin\left\{\frac{(2n+1)}{2}\pi z\right\}\times\exp\left\{-\frac{(2n+1)^2\pi^2}{4}\tau\right\} \tag{8}$$

For deep-pool systems, the Leveque-type solution can be obtained as follows [Carslaw and Jaeger, 1959]:

$$\theta_0=\tau\left[\left(1+\frac{1}{2}\eta^2\right)\mathrm{erf}\left(\frac{\eta}{2}\right)+\frac{1}{\sqrt{\pi}}\eta\ \exp\left(-\frac{1}{4}\eta^2\right)-\frac{1}{2}\eta^2\right] \tag{9}$$

where $\eta=z/\sqrt{\tau}$. Fig. 2 shows that the above equation is in accord with the exact solution of (8) in the short period of time ($\tau<0.1$). Therefore the solutions of (8) and (9) can be considered as exact ones for deep-pool systems. And the Eq. (9) suggest that the parameter $\eta(=z/\sqrt{\tau})$ play important role in analyzing the onset of thermal convection. Since we are primarily concerned with the deep-pool case of large Ra_I and small τ, the above Leveque type solution (9) represents the basic temperature profile quite well. But for the mathematical convenience in the present stability analysis we simplify the basic temperature profile by using the integral method [Eckert and Robert, 1972] as follows:

$$\theta_0=\tau[1-(1-\zeta)^3][1-U_{\zeta-1}] \tag{10}$$

where $\zeta=z/\delta\cdot U_{\zeta-1}$ is the unit step function having the zero value at $\zeta=1$ and δ is the dimensionless thermal penetration depth having the value of $\sqrt{8\tau}$. This approximate solution is in good agreement with the exact ones in the region of $\tau\leq0.1$, as shown in Fig. 2.

2. Stability Equations

Under the linear stability theory disturbances caused by the onset of thermal convection can be formulated, in dimensionless form, in terms of the temperature component θ_1 and the vertical velocity component w_1 by transforming Eqs. (1)-(3):

$$\left\{\frac{1}{Pr}\frac{\partial}{\partial\tau}-\bar{\nabla}^2\right\}\bar{\nabla}^2 w_1=-\bar{\nabla}_1{}^2\theta_1 \tag{11}$$

$$\frac{\partial\theta_1}{\partial\tau}+Ra_I w_1\frac{\partial\theta_0}{\partial z}=\bar{\nabla}^2\theta_1 \tag{12}$$

where $\bar{\nabla}^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and $\bar{\nabla}_1^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$. Here the velocity component has the scale of α/d and the temperature component has the scale of $\alpha\nu/(g\beta d^3)$. The proper boundary conditions are given by

$$w_1=Dw_1=\theta_1=0 \quad \text{at } z=0 \tag{13a}$$

$$w_1=Dw_1=D\theta_1=0 \quad \text{at } z=1 \tag{13b}$$

Our goal is to find the critical time τ_c for a given Pr and Ra_I by using Eqs. (11)-(13).

Based on the normal mode analysis, the amplitude functions w^* and θ^* are constructed as a function of $\zeta(=z/\delta)$ only by assuming periodic motion of disturbances in the form of regular cells over the horizontal plane:

$$[w_1(\tau,x,y,z),\ \theta_1(\tau,x,y,z)]=[\delta^2 w^*(\zeta),\ \theta^*(\zeta)]\ \exp[i(a_x x+a_y y)] \tag{14}$$

where "i" is the imaginary number. The horizontal wave number "a" has the relation of $a=[a_x^2+a_y^2]^{1/2}$ By using these relations the stability equation is obtained from equations (11)-(13) as

$$\left\{(D^2-a^{*2})^2+\frac{4}{Pr}(\zeta D^3-a^{*2}\zeta D+2a^{*2})\right\}w^*=-a^{*2}\theta^* \tag{15a}$$

$$(D^2+4\zeta D-a^{*2})\theta^*=Ra^* w^* D\theta_0^* \tag{15b}$$

where $a^*=a\delta$, $Ra^*=Ra_I\delta^3\tau$, $D=d/d\zeta$ and $\theta_0^*=\theta_0/\tau$. It is assumed that a^* and Ra^* are the eigenvalues, and also the onset time of buoyancy-driven convection for a given Ra_I is unique under the principle of exchange of stabilities. The above procedure is the essence of our propagation theory.

3. Solution Procedure

3-1. In the Case of Pr→∞

The stability equation derived in Eq. (15) still involves mathematical complexity. This problem can be alleviated by dealing with very high and very low Prandtl numbers. Let us consider the very high Pr case, first. Then the stability equations reduce to

$$(D^2+4\zeta D-a^{*2})(D^2-a^{*2})^2 w^*=-3a^{*2}Ra^*(1-2\zeta+\zeta^2)w^* \quad \text{for } 0\le\zeta\le 1 \tag{16a}$$

$$(D^2+4\zeta D-a^{*2})(D^2-a^{*2})^2 w^*=0 \quad \text{for } \zeta\ge 1 \tag{16b}$$

The above equations are separated, depending on the range of ζ. The boundary conditions can be converted to

$$w^*=Dw^*=(D^2-a^{*2})w^*=0 \quad \text{at } \zeta=0 \tag{17a}$$

$$w^*=Dw^*=D(D^2-a^{*2})w^*=0 \quad \text{at } \zeta=1/\delta\to\infty \tag{17b}$$

For a deep-pool system, the condition of $\zeta=1$ corresponds to the basic thermal penetration depth, and $1/\delta$ is practically equivalent to an infinite high value since δ is small.

Within the thermal penetration depth ($\zeta\le 1$) the velocity disturbance is approximated by means of rapidly converging power series proposed by Sparrow et al. [1964]:

$$w_i^*=\sum_{j=0}^{5}H_j f_j(\zeta) \tag{18a}$$

$$f_j(\zeta)=\sum_{n=0}^{\infty}b_n^{(j)}\zeta^n \tag{18b}$$

H_j (j=0, 1, 2, 3, 4, 5) is an arbitrary coefficient needed in the sixth-order differential equation, and $b_n^{(j)}$ can be obtained by substituting Eq. (18) into Eq. (16a) as the following indicial form:

$$\begin{aligned}b_n^{(j)}=\frac{(n-6)!}{n!}[&\{3a^{*2}(n-2)(n-3)(n-4)(n-5)\\&-4(n-2)(n-3)(n-4)(n-5)(n-6)\}b_{n-2}^{(j)}\\&+\{8a^{*2}(n-4)(n-5)(n-6)-3a^{*2}(n-4)(n-5)\}b_{n-4}^{(j)}\\&+\{a^{*2}-4^{*2}(n-6)-3a^{*2}Ra^*\}b_{n-6}^{(j)}+6a^{*2}Ra^* b_{n-7}^{(j)}\\&-3a^{*2}Ra^* b_{n-8}^{(j)}]\end{aligned} \tag{19a}$$

$$b_n^{(j)}=\delta_{n,j}\ (n=0,1,2,3,4,5):\text{Kronecker delta} \tag{19b}$$

$$b_{-1}^{(j)}=b_{-2}^{(j)}=b_{-3}^{(j)}=b_{-4}^{(j)}=b_{-5}^{(j)}=b_{-6}^{(j)}=b_{-7}^{(j)}=b_{-8}^{(j)}=0 \tag{19c}$$

Applying the boundary condition of $\zeta=0$ to Eqs. (18) and (19), the velocity disturbances inside the thermal penetration depth, $\zeta\le 1$, can be expressed in the following form:

$$w_i^*=H_2\left\{f_2(\zeta)+\frac{a^{*2}}{6}f_4(\zeta)\right\}+H_3 f_3(\zeta)+H_5 f_5(\zeta) \tag{20}$$

In order to obtain the velocity disturbance for the region of $\zeta\ge 1$, it is helpful to consider the solution outside the thermal penetration depth in two stages:

$$(D^2+4\zeta D-a^{*2})Y=0 \tag{21a}$$

$$(D^2-a^{*2})^2 w_o^*=Y \tag{21b}$$

The WKB method can be used to obtain the solution of Y which satisfies the condition DY=0 as $\zeta\to\infty$. Then the solution of Y is given by [Mathews and Walker, 1973]:

$$Y\cong\frac{\exp(-\zeta^2)}{4\sqrt{4\zeta^2+2+a^{*2}}}\exp\left\{-\int_1^{\zeta}\sqrt{4\xi^2+2+a^{*2}}\,d\xi\right\} \tag{22}$$

The form of Y as is determined by the WKB method is very complicated. In order to find the particular solution of Eq. (21b) over the range of $\zeta\ge 1$, ζ is converted as

$$s=\zeta-1 \tag{23}$$

which provides the convergence in computer calculation. By using the initial values of Y and DY at $\zeta=1$ the solution of Y is obtained in form of power series. The solution of the velocity disturbance outside the thermal penetration depth can be obtained by inverse-operator technique, as follows:

$$\begin{aligned}w_o^*=&H_7\exp(-a^*s)+H_9 s\exp(-a^*s)\\&+\frac{H_{10}}{4a^{*2}}\left\{\exp(a^*s)\sum_{n=0}^{\infty}\frac{p_n}{(n+1)(n+2)}s^{n+2}\right.\\&+\exp(-a^*s)\sum_{n=0}^{\infty}\frac{q_n}{(n+1)(n+2)}s^{n+2}\\&-\frac{1}{a^*}\exp(a^*s)\sum_{n=0}^{\infty}\frac{p_n}{(n+1)}s^{n+1}\\&\left.+\frac{1}{a^*}\exp(-a^*s)\sum_{n=0}^{\infty}\frac{q_n}{(n+1)}s^{n+1}\right\}\end{aligned} \tag{24a}$$

$$p_n=-\frac{(n-2)!}{n!}\{(2a^*+4)(n-1)p_{n-1}+4(n-2+a^*)p_{n-2}+4a^*p_{n-3}\} \tag{24b}$$

$$p_2=-\frac{1}{2}\{(2a^*+4)p_1+4a^*p_0\} \tag{24c}$$

$$p_1=Y'(1)-a^*Y(1) \tag{24d}$$

$$p_0=Y(1) \tag{24e}$$

$$q_n = -\frac{(n-2)!}{n!}\{(4-2a^*)(n-1)q_{n-1} + 4(n-2-a^*)q_{n-2} - 4a^*q_{n-3}\} \quad (24f)$$

$$q_2 = -\frac{1}{2}\{(4-2a^*)q_1 - 4a^*q_0\} \quad (24g)$$

$$q_1 = Y'(1) + a^*Y(1) \quad (24h)$$

$$q_0 = Y(1) \quad (24i)$$

The disturbance Eqs. (22) and (24) for velocities both inside and outside the thermal penetration depth are patched at $\zeta=1$ where the velocity, the stress and the temperature are all continuous in a physical sense. Mathematically, the expression for the velocity disturbance is an analytical function at the thermal penetration depth. Thus the following relations must be satisfied:

$$D^n w_i^* = D^n w_o \ (n=0,\ 1,\ 2,\ 3,\ 4,\ 5) \text{ at } \zeta=1 \quad (25)$$

The above relations can be expressed in matrix form as

$$\begin{bmatrix} f_2+(a^{*2}/6)f_4 & f_3 & f_5 & -1 & 0 & 0 \\ Df_2+(a^{*2}/6)Df_4 & Df_3 & Df_5 & a^{*2} & -1 & 0 \\ D^2f_2+(a^{*2}/6)D^2f_4 & D^2f_3 & D^2f_5 & -a^{*2} & 2a^* & 0 \\ D^3f_2+(a^{*2}/6)D^3f_4 & D^3f_3 & D^3f_5 & a^{*3} & -3a^{*2} & 0 \\ D^4f_2+(a^{*2}/6)D^4f_4 & D^4f_3 & D^4f_5 & -a^{*4} & 4a^{*3} & -Y \\ D^5f_2+(a^{*2}/6)D^5f_4 & D^5f_3 & D^5f_5 & a^{*5} & -5a^{*4} & -Y' \end{bmatrix}_{\zeta=1} \times \begin{bmatrix} H_2 \\ H_3 \\ H_5 \\ H_7 \\ H_9 \\ H_{10} \end{bmatrix} = 0 \quad (26)$$

To produce nontrivial solution of velocity disturbances, the determinant of 6×6 matrix must be zero. The value of the determinant is determined by the two eigenvalue a^* and Ra^*. Therefore the computer calculation was carried out to obtain Ra^* for a given a^*.

3-2. In the Case of $Pr\to0$

Stability analysis for the very small Prandtl number case is basically similar to the case of $Pr\to\infty$. In the limiting case of $Pr\to0$, however, the viscous effects of amplitude function can be ignored in comparison to the convective effects. Also, boundary conditions should be relaxed under the approximation of negligible viscous effects. Therefore, the no-slip boundary conditions cannot be applied at $\zeta=0$. The resulting stability equations and their boundary conditions are reduced as follows:

$$(D^2+4\zeta D-a^{*2})(\zeta D^3-a^{*2}\zeta D+2a^{*2})w^* = -\frac{3}{4}PrRa^*a^{*2}w^*(1-\zeta)^2 \quad \text{for } 0\le\zeta\le1 \quad (27a)$$

$$(D^2+4\zeta D-a^{*2})(\zeta D^3-a^{*2}\zeta D+2a^{*2})w^* = 0 \quad \text{for } \zeta\ge1 \quad (27b)$$

with boundary conditions

$$w^* = \theta^* = 0 \quad \text{at } \zeta=0 \quad (28a)$$

$$w^* = Dw^* = D\theta^* = 0 \quad \text{as } \zeta\to\infty \quad (28b)$$

The inner solution can not be easily obtained as the rapidly converging power series form because of the non-linear characteristic of convective term. Thus Frobenius method is applied in this study as follows:

$$w_i^* = \sum_{n=0}^{\infty} b_n\zeta^{n+c} \quad (29)$$

Substitution of Eq. (29) into (27a) makes the following indicial equation.

$$c(c-1)(c-2)^2(c-3) = 0 \quad (30)$$

Now, we can outline the form of the solution for the each induce "c" and obtain the solution as 5 independent series.

$$\begin{aligned} w_i^* = {} & G_0\left\{1-\frac{1}{48}\left(\frac{3}{4}PrRa^*a^{*2}-2a^{*2}\right)\zeta^4+\cdots\right\} \\ & +G_1\left\{\zeta-\frac{1}{360}\left(\frac{3}{4}PrRa^*a^{*2}+10a^{*2}-a^{*4}\right)\zeta^5+\cdots\right\} \\ & +G_2\left\{\zeta^2-\frac{1}{1440}\left(\frac{3}{4}PrRa^*a^{*2}\zeta^6\right)+\cdots\right\} \\ & +G_3\left\{\zeta^3-\frac{1}{30}(2-a^{*2})\zeta^5+\cdots\right\} \\ & +G_4\left\{\left(\zeta^2-\frac{1}{1440}\frac{4}{3}PrRa^*a^{*2}\zeta^6+\cdots\right)\ln\zeta+\left(\frac{a^{*2}}{12}\zeta^4+\cdots\right)\right\} \end{aligned} \quad (31)$$

where coefficients G_i ($i=0,\ 1,\ 2,\ 3,\ 4$) are arbitrary constants. Under the boundary conditions that the velocity and temperature perturbations do not exist at the rigid-isothermal surface, G_0 and G_4 are to be eliminated.

The outer solution in the infinite domain can be obtained by separation Eq. (27b) into

$$(D^2+4\zeta D-a^{*2})Y = 0 \quad (32a)$$

$$(D^2-a^{*2})(\zeta D-2)w_o^* = Y \quad (32b)$$

The asymptotic solution of Eq. (32a) is the same as Eq. (22). By using this solution, we can obtain the outer amplitude function similar to the previous case. As the first step, the homogeneous solution of Eq. (32b) can be produced as

$$w_{o,h}^* = \frac{G_5}{2}\left\{(1-a^*\zeta)\ \exp(-a^*\zeta)+a^{*2}\zeta^2\int_\zeta^\infty\frac{\exp(-a^*\xi)}{\xi}d\xi\right\} \quad (33)$$

And, Eq. (32a) is transformed into those of $s=\zeta-1$. Then the solution of Y is generated as the forms of $\exp(a^*s)p(s)$ and $\exp(-a^*s)q(s)$. p(s) and q(s) are the power series forms as the function of s, whose coefficients are dependent of the asymptotic solution. Consequently Eq. (32b) can be written through the operator technique as

$$\{(s+1)D^*-2\}w_o^* = \frac{G_6}{2a^*}\left\{\exp(a^*s)\sum_{n=0}^{\infty}\frac{p_n}{n+1}s^{n+1} - \exp(-a^*s)\sum_{n=0}^{\infty}\frac{q_n}{n+1}s^{n+1}\right\} \quad (34)$$

with $p_0=q_0=Y(1)$, $p_1=Y'(1)-a^*Y(1)$, and $q_1=Y'(1)+a^*Y(1)$. For $n\ge2$, the recursion formula for p_n and q_n can be easily constructed, and are identical with Eq. (24). The particular solution is obtained in the form of

$$w_{o,p}^* = \exp(a^*s)\sum_{n=0}^{\infty}d_ns^n+\exp(-a^*s)\sum_{n=0}^{\infty}e_ns^n \quad (35a)$$

$$d_0 = d_1 = 0 \quad (35b)$$

$$d_2 = \frac{G_6}{2a^*}\frac{p_0}{2} \quad (35c)$$

$$d_3 = \frac{G_6}{2a^*}\frac{1}{6}\{p_1-a^*p_0\} \quad (35d)$$

$$d_{n+2} = \frac{1}{n+2}\left\{\frac{G_6}{2a^*}\frac{p_n}{n+1}-(n-1+a^*)d_{n+1}-a^*d_n\right\} \text{ for } n\ge2 \quad (35e)$$

$$e_0 = e_1 = 0 \quad (35f)$$

$$e_2 = -\frac{G_6}{2a^*}\frac{q_0}{2} \quad (35g)$$

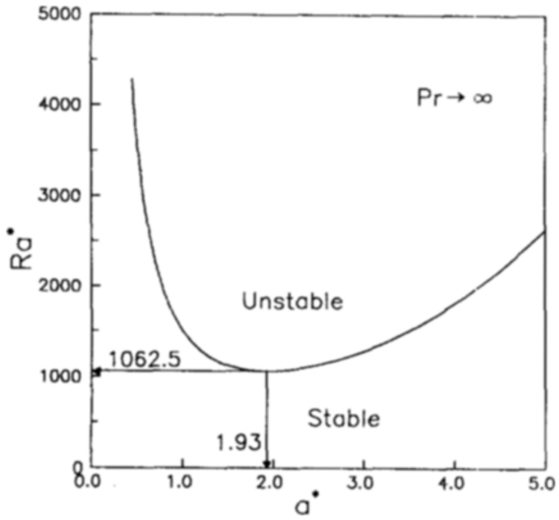

Fig. 3. Neutral stability curve for the infinite Prandtl number.

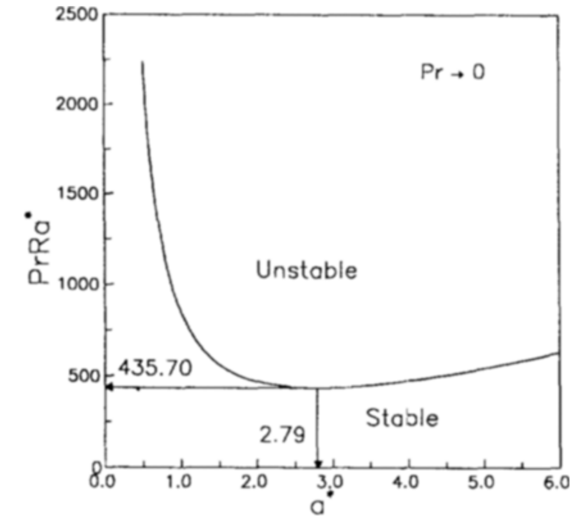

Fig. 4. Neutral stability curve for the zero Prandtl number.

$$e_3 = -\frac{G_6}{2a^*}\frac{1}{6}\{q_1 - a^* q_0\} \tag{35h}$$

$$e_{n+2} = -\frac{1}{n+2}\left\{\frac{G_6}{2a^*}\frac{q_n}{n+1} - (n-1+a^*)e_{n+1} - a^* e_n\right\} \quad \text{for } n \geq 2 \tag{35i}$$

The outer solution can be obtained as $w_o^* = w_{o,h}^* + w_{o,p}^*$. Since the solution to satisfy all the boundary conditions are found in the whole domain, the following equation to characterize the onset of convection is generated by using Eq. (25) as the previous case:

$$\begin{bmatrix} G^{(1)} & G^{(2)} & G^{(3)} & -(e^{-a^*} - e^{a^*} + a^{*2}E_i)/2 & 0 \\ DG^{(1)} & DG^{(2)} & DG^{(3)} & a^* e^{-a^*} - a^{*2}E_i & 0 \\ D^2G^{(1)} & D^2G^{(2)} & D^2G^{(3)} & -a^{*2}E_i & 0 \\ D^3G^{(1)} & D^3G^{(2)} & D^3G^{(3)} & a^{*2}e^{-a^*} & -Y \\ D^4G^{(1)} & D^4G^{(2)} & D^4G^{(3)} & -(a^{*3}+a^{*2})e^{-a^*} & -Y'+Y \end{bmatrix}_{\zeta=1} \times \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_5 \\ G_6 \end{bmatrix} = 0 \tag{36}$$

where $G^{(i)}$ (i=0, 1, 2, 3, 4) is a infinite series with respect to G_i in Eq. (31) and $E_i = \int_\zeta^\infty \{\exp(-a^*\xi)/\xi\}/d\xi$. The value of E_i can be obtained by using IMSL subroutine library. PrRa* results from the condition that the determinant of resulting 5×5 square matrix is equal to zero. The minimum value of PrRa* in the plot of PrRa* vs. a* is the critical condition to mark the onset of natural convection for extremely small Prandtl number.

STABILITY RESULTS

The marginal stability curves obtained from computer calculation are shown in Figs. 3 and 4. And the critical condition for the onset of buoyancy-driven convection are

$$Ra_c^* = 1062.50 \text{ and } a_c^* = 1.93 \qquad \text{for } Pr \to \infty \tag{37a}$$

$$PrRa_c^* = 435.70 \text{ and } a_c^* = 2.79 \qquad \text{for } Pr \to 0 \tag{37b}$$

From the aboves, onset time τ_c are expressed as

$$\tau_c = 4.66 Ra_l^{-2/5} \qquad \text{for } Pr \to \infty \tag{38a}$$

$$\tau_c = 3.26 (PrRa_l)^{-2/5} \qquad \text{for } Pr \to 0 \tag{38b}$$

Based on the results for the limiting cases, the stability criteria for a deep-pool system may be roughly constructed as

$$Ra_c^* = 1062.50\left(1 + \frac{0.41}{Pr}\right) \tag{39}$$

Therefore, the onset time of buoyancy-driven convection may have the following relation:

$$\tau_c = 4.66\left(1 + \frac{0.41}{Pr}\right)^{2/5} Ra_l^{-2/5} \quad \text{for } \tau_c \leq 0.1 \tag{40}$$

Fluid properties of in term of the Prandtl number have a profound effect on the stability conditions; smaller Prandtl number fluids are more stable due to dominant conducting effects.

Foster [1969] proposed that the onset time of natural convection obtained by using the thermal penetration depth as a length scaling factor should be shortened by factor of 4. Considering Foster's concept, we suggest that the disturbances set in at τ_c will manifest themselves around $4\tau_c$. Thus, we foretell the onset time when the convective motion can be detectable experimentally, τ_o, as follows:

$$\tau_o = 18.64\left(1 + \frac{0.41}{Pr}\right)^{2/5} Ra_l^{-2/5} \tag{41}$$

The relationship $\tau_o = 4\tau_c$ can be seen in many other systems [Yoon and Choi, 1989; Choi et al., 1988].

HEAT TRANSPORT

The possibility of connecting the stability criteria to fully-developed turbulent thermal convection has been discussed by Howard [1964]. According to Howard's concept called the boundary-layer instability model, the heat transport in fully-developed turbulent state is governed by the narrow region of the heated surface for systems heated isothermally from below. Its modification extending Howard's concept is shown in Fig. 5.

From the boundary layer instability model the Nusselt number, $Nu = Sd^2/k\Delta T$ can be expressed as follows:

$$Nu = \frac{d}{\delta^*} \qquad \text{for } Ra_l \to \infty \tag{42}$$

where δ^* is the conduction thickness. This may be replaced by

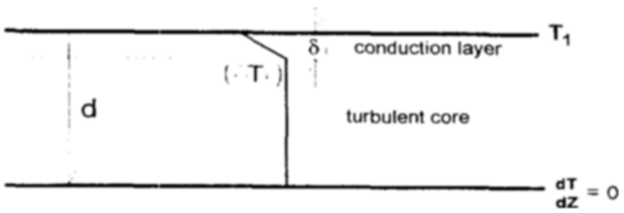

Fig. 5. Simple diagram of turbulent heat transport model.

δ_c following Howard's concept where δ_c is thermal penetration depth at the onset condition of buoyancy-driven convection. Thus, by using the relation of $Ra_I = RaNu$ Eq. (42) can be expressed as

$$Nu = \left(\frac{Ra_I}{Ra_{\delta_c}}\right)^{1/4} \quad \text{for } Ra_I \to \infty \tag{43}$$

where Ra_{δ_c} is represented by

$$Ra_{\delta_c} = \frac{g\beta\delta_c^{\,3}\Delta T|_\delta}{\alpha\nu} \tag{44}$$

$\Delta T|_\delta$ is the temperature difference across the boundary layer and can be expressed as

$$\Delta T|_\delta = \frac{S\tau}{k} \tag{45}$$

From the Eqs. (44) and (45) Ra_{δ_c} can be substituted by Ra^*. Then the heat transport in the fully-developed state is governed by

$$Nu = \frac{0.1752 Ra_I^{1/4}}{(1+0.41/Pr)^{1/4}} \quad \text{for } Ra_I \to \infty \tag{46}$$

Arpaci [1994] proposed the backbone equations to predict the heat transport for the horizontal fluid layer heated below. By modifying the Arpaci's results, a new backbone equation to govern the buoyancy-driven heat transport in the present system can be obtained as follows:

$$Nu = 2 + \frac{A\left(\frac{Pr}{C+Pr}\right)^{1/4}(Ra_I^{1/4} - Ra_{I,c}^{1/4})}{\left[1 - B\left(\frac{Pr}{C+Pr}\right)^{-1/9}(Ra_I/Nu)^{-1/9}\right]^{3/4}} \quad \text{for } Ra_I \geq Ra_{I,c} \tag{47}$$

where A, B and C are the undetermined constants. It is noted that $Nu = 2$ for $Ra_I \leq Ra_{I,c} (=2772)$, where $Ra_{I,c}$ is the critical Rayleigh number to mark the onset of buoyancy-driven convection in the present system. From the result of heat transfer relation of infinite Rayleigh number case, A and C can be determined easily. For the infinite Rayleigh number, the values of A and C in Eq. (47) can be determined from Eq. (46) as 0.1752 and 0.41, respectively.

The finite-amplitude heat transfer characteristics slightly over $Ra_{I,c}$ can be obtained by using the shape assumption of Stuart [1964]. For the region of $Ra_I \to Ra_{I,c}$, Roberts [1967] expressed the Nusselt number as

$$\frac{2}{Nu} = 1 - \frac{\Gamma}{Ra_I}(Ra_I - Ra_{I,c}) \tag{48}$$

The constant Γ is obtained from the distribution of disturbance quantities at $Ra = Ra_c$:

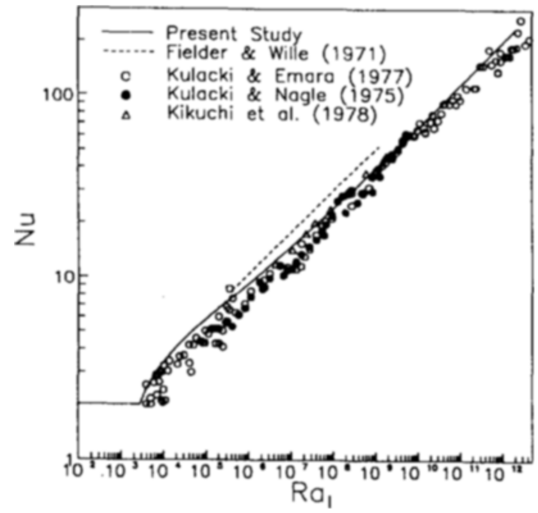

Fig. 6. Predicted heat transfer correlation and existing experimental data.

$$\Gamma = \frac{2\int_0^1 z w_1\theta_1 dz \int_0^1 w_1\theta_1 dz}{\int_0^1 (w_1\theta_1)^2 dz} = 0.5994 \tag{49}$$

Thus from the Eqs. (48) and (49), we obtain the following relation:

$$\left.\frac{dNu}{dRa_I}\right|_{Ra_I \to Ra_{I,c}} = \frac{2\Gamma}{Ra_{I,c}} \tag{50}$$

The above relations have its meaning for large Prandtl number.

It is interesting that only one initial slope of Nu with respect to Ra_I is necessary to determine the numerical value of B. Assembling the Eq. (46), (47) and (50), we can derive a new heat transfer correlation for the whole range of Ra_I as:

$$Nu = 2 + \frac{0.1752(Ra_I^{1/4} - 2772^{1/4})}{(1+0.41/Pr)[1 - 1.8535(1+0.41/Pr)^{1/9}(Ra_I/Nu)^{-1/9}]^{3/4}} \quad \text{for } Ra_I \geq 2772 \tag{51}$$

The proposed correlation covering the whole range of Pr is compared with the experimental results of water. The above equation for water ($Pr = 7$) predicts the heat transport quite well, as shown in Fig. 6.

CONCLUSION

The onset of regular cell-type motion in a horizontal fluid layer with uniform volumetric energy sources has been studied analytically by using linear stability theory. Our propagation theory predicts that the onset time of buoyancy driven motion is a function of the Rayleigh number and Prandtl number. Also, based on the boundary-layer instability model and Arpaci's model, heat transfer characteristics of the layer are predicted as a function of the Rayleigh number and Prandtl number. These results show that the propagation theory we have developed is a powerful tool in analyzing buoyancy-driven phenomena.

REFERENCES

Arpaci, V. S., "Microscales of Turbulence Heat Transfer Correlations", Proc. 10th Int. Heat Transfer Conf., Brighton, U.K., **1**, 129 (1994).

Benard, H., "Les Tourbillons Celluaries dan une nappe Liquid Transportant de la chaleur par Convection en Regime Permant", *Ann. Chem. Phys.*, **23**, 62 (1901).

Berg, J. C., Acrivos, A. and Boudart, M., "Evaporative Convection", *Adv. Chem. Eng.*, **6**, 61 (1974).

Carslaw, H. S. and Jaeger, J. C., "Conduction of Heat in Solids", Oxford Univ. Press, England (1959).

Chandrasekhar, S., "Hydrodynamic and Hydromagnetic Stability", Oxford Univ. Press, London (1961).

Cheung, F. B., "Heat Source-Driven Thermal Convection at Arbitrary Prandtl Number", *J. Fluid Mech.*, **97**, 734 (1980).

Choi, C. K. and Kim, M. C., "Convective Instability in the Thermal Entrance Region of Plane Couette Flow Heated Uniformly from Below", Proc. 9th Int. Heat Transfer Conf., Jerusalem, **2**, 519 (1990).

Choi, C. K., Lee, J. D., Hwang, S. T. and Yoo, J. S., "The Analysis of Thermal Instability and Heat Transfer Prediction in a Horizontal Fluid Layer Heated from Below", Frontiers of Fluid Mech. (ed. by Shen Yuen), Pergamon Press, Oxford, 1193 (1988).

Choi, C. K., Shin, C. B. and Hwang, S. T., "Thermal Instability in Thermal Entrance Region of Plane Couette Flow Heated Uniformly from Below", Proc. 8th Int. Heat Transfer Conf., San Francisco, **3**, 1389 (1984).

Chun, Y. H. and Choi, C. K., "Thermal Instability of Natural Convection over Inclined Isothermally Heated Plate", *HWAHAK KONGHAK*, **29**, 381 (1991).

Eckert, E. R. G. and Robert, M. D., "Analysis of Heat and Mass Transfer", McGraw Hill, New York (1972).

Fidler, H. and Wille, R., "Warmetransport bie fier Konvection in einer horizontalen Flussigkeitsschicht mit Volumenheizung", Teil 1, Rep. Deutsch Forschungs Varsuchsantsalt Luft-Raumfahrt, Inst. Turbulenzfenschung, Berlin (1971).

Foster, T. D., "Stability of a Homogeneous Fluid Cooled Uniformly from Below", *Phys. Fluids*, **8**, 1249 (1965).

Foster, T. D., "Onset of Manifest Convection in a Layer of Fluid with a Time-Dependent Surface Temperature", *Phys. Fluids*, **12**, 2482 (1969).

Howard, L. N., "Convection at High Rayleigh Number", Proc. 11th Int. Congress Appl. Mech., Munich, 1109 (1964).

Jhaveri, B. S. and Homsy, G. M., "The Onset of Convection in Fluid Layer Heated Rapidly in a Time-Dependent Manner", *J. Fluid Mech.*, **114**, 251 (1982).

Kikuchi, Y., Kawasaki, T. and Shioyama, T., "Thermal Convection in a Horizontal Fluid Layer Heated Internally and from Below", *Int. J. Heat Mass Transfer*, **25**, 363 (1978).

Kim, M. C., Choi, C. K. and Davis, E. J., "Thermal Instability of Blasius Flow over Isothermally Heated Horizontal Plates", *Int. J. Eng. Fluid Mech.*, **3**, 71 (1990).

Kim, M. C. and Choi, C. K., "Buoyancy Effects in Thermally Developing Plane Poiseuille Flow", Proc. 5th Asian Cong. Fluid Mech., Taejon, **1**, 310 (1992).

Kulack, F. A. and Emara, A. A., "Steady and Transient Thermal Convection in a Fluid Layer with Uniform Volumetric Energy Sources", *J. Fluid Mech.*, **83**, 375 (1977).

Kulacki, F. A. and Goldstein, R. J., "Hydrodynaimc Stability in Fluid Layer with Volumetric Energy Sources", *Appl. Sci. Res.*, **13**, 81 (1975).

Kulacki, F. A. and Nagle, M. E., "Natural Convection in a Horizontal Fluid Layer with Volumetric Energy Sources", *J. Heat Transfer*, **97**, 204 (1975).

Long, R. R., "The Relation between Nusselt Number and Rayleigh Number in Turbulent Thermal Convection", *J. Fluid Mech.*, **73**, 445 (1976).

Lord Rayleigh, "On Convection Current in a Horizontal Layer of Fluid when the Higher Temperature is on the Under Side", *Philos. Mag.*, **32**, 529 (1916).

Mathews, J. and Walker, R. L., "Mathematical Method of Physics", Benjamin, California (1973).

Roberts, P. H., "Convection in Horizontal Layer with Internal Heat Generation", *J. Fluid Mech.*, **30**, 30 (1967).

Sparrow, E. M., Goldstein, R. J. and Jonsson, V. K., "Thermal Instability in a Horizontal Fluid Layer : Effects of Boundary Conditions and Non-linear Temperature Profile", *J. Fluid Mech.*, **18**, 513 (1964).

Stuart, J. T., "On Cellular Patterns in Thermal Convection", *J. Fluid Mech.*, **18**, 481 (1964).

Wankat, P. C. and Homsy, G. M., "Lower Bounds for the Onset Time of Instability in Heated Layer", *Phys. Fluids*, **20**, 1200 (1977).

Yoon, D. Y. and Choi, C. K., "Thermal Convection in a Saturated Porous Medium Subjected to Isothermal Heating", *Korean J. Chem. Eng.*, **6**, 144 (1989).